

NOTE

**GARSIA AND MILNE'S BIJECTIVE PROOF OF THE
INCLUSION-EXCLUSION PRINCIPLE**

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Although the following proof is implicit in Garsia and Milne's paper [1], it is so elegant that we felt that it should be presented by itself for the benefit of the general mathematical public. The idea behind the proof was further exploited by Remmel [2] and Wilf [3].

Consider a set A of elements each of which possess a (possibly empty) subset of the properties $\{1, \dots, n\}$. The inclusion-exclusion principle states that the number of elements with no properties is

$$\sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|} |A_I|. \quad (1)$$

Here, for any subset I of $\{1, \dots, n\}$, A_I denotes the set of elements having all the properties of I and, for any set B , $|B|$ denotes the number of elements of B .

Our proof starts by introducing the much larger set \mathcal{A} of all possible pairs (a, J) where a is an element of A and J is a subset of the set of properties of a . The pair (a, J) is *even* or *odd* according to whether $|J|$ is even or odd respectively. We next observe that for a fixed $I \subseteq \{1, \dots, n\}$, (a, I) is a legitimate pair if and only if $a \in A_I$. It follows that (1) expresses the difference between the number of even and odd pairs.

For any $a \in A$ let $s(a)$ be its smallest property. Define the following mapping from \mathcal{A} to itself:

$$T(a, J) = \begin{cases} (a, J \cup s(a)), & s(a) \notin J, \\ (a, J/s(a)), & s(a) \in J. \end{cases}$$

This is a parity changing involution which is defined everywhere *except* on pairs of the form (a, \emptyset) where a is a property-less element of A . It follows that the odd pairs of \mathcal{A} are equinumerous with the even pairs of \mathcal{A} which are *not* of the above form. This implies (1) since the pairs (a, \emptyset) for which a is property-less are obviously equinumerous with the set of property-less elements of A .

Our proof readily extends to prove the following generalized version of the inclusion-exclusion principle. Let t_1, \dots, t_n be commuting indeterminates and for $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ denote $t_I = t_{i_1} \cdots t_{i_k}$ and $(t-1)_I = (t_{i_1} - 1) \cdots (t_{i_k} - 1)$, then if for $a \in A$, $\text{Prop}(a)$ denotes the set of properties of a then

$$\sum_{a \in A} t_{\text{prop}(a)} = \sum_{I \subset \{1, \dots, n\}} |A_I| (t-1)_I.$$

References

- [1] A.M. Garsia and S.C. Milne, A Rogers-Ramanujan bijection, *J. Combin. Theory Ser. A* 31 (1981) 289-339.
- [2] J. Remmel, Bijective proofs of some classical partition identities, to appear.
- [3] H.S. Wilf, Sieve equivalence in generalized partition theory, *J. Combin. Theory Ser. A* 34 (1983) 80-89.